7. TROITSKII V.A. and PETUKHOV L.V., Optimization of Shapes of Elastic Bodies. Nauka, Moscow, 1982.
8. OLHOFF N., Optimal Design of Structures /Russian translation/ Mir, Moscow, 1981.
9. LUR'E K.A. and CHERKAEV A.V., On application of the Prager theorem to the problem of optimal thin plate design, Izv. Akad. Nauk SSSR, Mekhan. Tverd Tela, 6, 1976.
10. LITVINOV V.G., Optimal control problem for the natural frequency of a plate of variable thickness, Zh. Vychisl. Mat. Matem. Fiz., 19, 4, 1979.
11. LITVINOV V.G., Optimal control of the coefficients in elliptic systems, Differents. Uravneniia, 6, 1982.
12. MIKHLIN S.G., Variational Methods in Mathematical Physics, Nauka, Moscow, 1970.
13. KANTOROVICH L.V. and AKILOV G.P., Functional Analysis, Nauka, Moscow, 1977.
14. KOHN R.V. and VOGELIUS M., A new model for thin plates with rapidly varying thickness, I. Proc. Univ. Maryland, 988, August, 1982.
15. SAMSONOV A.M., Optimal location of a thin elastic rib on an elastic plate, Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, 1, 1978.
16. COURANT R. and HILBERT D., Methods of Mathematical Physics, Vol.1 /Russian translation/, Gostekhteorizdat, Moscow-Leningrad, 1933.
17. REISZ F. and SZEKEFALVY-NAGY B., Lectures on Functional Analysis /Russian translation/, Mir, Moscow, 1979.
18. ALEKSEYEV V.M., TIKHOMIROV V.M. and FOMIN S.V., Optimal Control, Nauka, Moscow, 1979.
19. TIMOSHENKO S.P. and WOINOWSKI-KRIEGER S., Plates and Shells /Russian translation/, Fizmatgiz, Moscow, 1963.

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STOCHASTIC BIFURCATION IN THE THEORY OF THE FLEXURE OF SPHERICAL SHELLS AND CIRCULAR MEMBRANES*

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The capacity of rigidly clamped elastic membranes and open shallow spherical shells of circular outline that are in equilibrium under the action of a radial stress, given uniformly on the contour, and transverse loads distributed radially along the surface to form a field with a quasi-Gaussian probability measure to retain shape is investigated. It is assumed that the behaviour of the membranes and shells is described by von Karman equations taken in a radial approximation.

The following method /1/ is used. A generalization of the probability density, a probability functional (PF) induced by the probability measure of the load and the operator of the problem is constructed in the space of possible solutions of the initial boundary value problem (the concept of probability density inthe functional space of individual realizations of a random field of the desired parameters was first utilized in statistical hydromechanics problems /2/). The times of a substantial change in the shape or an abrupt decrease in the shell (and membrance) carrying capacity are related to the first bifurcation of the PF modes with respect to the growth of the compressive force.

The application of this method starts with the derivation of the equations for the PF extremals in the space of weighted derivatives of the deflection function with respect to the dimensionless variable radius. Within the framework of the Galerkin method, solutions of the designated equation are determined. Simple relationships are determined that relate the radial stresses to the statistical characteristics of the transverse load field at the time of bifurcation of these solutions. It is shown that up to the time of the first bifurcation of PF has just one extremal, a trivial mode for the membranes but a non-trivial mode for the shells. Then by starting with the time mentioned the membrane PF reaches maxima on the extremals bifurcating from the trivial, while the shell PF acquires a new maximum (in addition to the existing maximum) on still another

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non-trivial extremal. Results are presented of a computation of the compressive forces of the first bifurcation of the PF modes in the case of transverse loads with a small correlation scale.

1. We consider the axisymmetric elastic strain of an open, rigidly clamped, shallow spherical shell that is in equilibrium under the action of a uniform radial stress on the contour and a radially distributed transverse load on the surface. The behaviour of such a shell is described by von Karman equations taken in a radial approximation /3/. Let the radii of the reference contour a and of the sphere R be in the ratio $\varepsilon = a/R \ll 1$. Then in the dimensionless von Karman equations solved for the stress function, the terms of second and higher order in ε can be neglected. We consequently arrive at the following set of equations

$$N[r;u] = f(r) \tag{1.1}$$

$$N[r;u] = Lu(r) + \alpha N_1[r;u] + \alpha\varepsilon N_2[r;u], \quad L = \frac{d^2}{dr^2} + \lambda - \frac{3}{4r^2}$$

$$N_1[r;u] = \int_0^1 ds g(r,s)(rs)^{-1/2} u^2(s) u(r)$$

$$N_2[r;u] = \int_0^1 ds g(r,s)(2r^{-1/2}u(r) + s^{-1/2}u(s)) u(s)$$

$$u = r^{1/2}\frac{dv}{dr}, \quad f(r) = r^{-1/2}\int_0^r ds\pi(s)$$

$$|r^{-1/2}u|_{r=0} < \infty, \quad u|_{r=1} = 0 \quad (v|_{r=1} = 0) \tag{1.2}$$

The dimensionless quantities in (1.1) and (1.2) are related to the dimensional formulas

$$a\{r, v, z\} = \{\rho, V, Z\}, \; Da^{-2}\{h\sigma_\rho, q\}$$

$$\alpha = (aR_*^{-1})^2, \; R_*^2 = 2D(hE)^{-1}, \; 0 \leqslant \varepsilon \ll 1$$

Here V is the deflection of points of the middle surface Z, ρ is a variable radius, E is Young's modulus, h is the shell thickness, D is the cylindrical stiffness, σ_ρ is the radial stress on the contour, q is the transverse radially distributed load, $g(r, s)$ is Green's function of problem (1.1), (1.2) for $\alpha = \lambda = 0$, and writing $\Phi[r; u]$ means that the quantity Φ is a function of r and a functional of the field $u(r)$.

Let Ω be the space of elementary events ω, U the set in which solutions of (1.1) are sought with the conditions (1.2), and F a set dependent on the case of the right sides of (1.1). We assume that the mapping $\omega \to f(|\omega)$ is characterized by a non-negative normalized Borel measure $\mu_f(\omega)$ given on Ω such that higher moments of the random field $f(r)$ are expressed in terms of the lowest by a method close to a Gaussian field. The question is what elements of of U possess the greatest probability for $\lambda > 0$ and for what values of λ does bifurcation of these elements occur.

2. We introduce a set of trial functions Φ including, in particular, the orthonormalized eigenfunctions φ_j of problem (1.1), (1.2) linearized for $f = 0$ in the segment $[0, 1]$.
We set

$$L[u, f, \varphi] = \int_0^1 dr\,(N[r;u] - f(r))\,\varphi(r)$$

We extract the subset Γ of all pairs $(f, u) \in F \times U$ in $F \times U$ that satisfy the condition

$$L[u, f, \varphi] = 0, \quad \forall\varphi \in \Phi \tag{2.1}$$

We let the symbol $u(r|\omega)$ (or $u[r; f(|\omega)]$)denote the stochastic solution of the initial problem which we shall understand to be a Borel measurable mapping $\Omega \to U$ (with a value at the point ω) that satisfies the system (2.1), (1.2) and corresponds to the mapping $\omega \to f(r|\omega)$: $\Omega \to F$ (the existence of such solutions follows from the proof of Theorem 1 in /4/).

Let M be the projection of the space Γ on U and $C(F \times M)$ the Banach space of all complex-valued functionals $\psi[f, u]$ continuous $F \times M$ with natural definition of the norm

$$\|\psi\| = \sup|\psi[f, u]|, \quad \forall\psi \in C(F \times M)$$

Then under the conditions of this problem, in $F \times M$ there exists /4/ a normalized non-negative Radon measure μ^T connected with $\mu_f(\omega)$ by the relationship

$$\int_\Omega d\mu_f(\omega)\psi[f(|\omega), u(|\omega)] = \int_{F\times M} d\mu^T\psi[f, u] \tag{2.2}$$

According to the remarks of Theorem 2 in /4/, the Lebesgue extension /5/ of the measure

μ^T satisfies the conditions

$$\int_{F\times M} d\mu^T \psi[f,u] L[u,f,\varphi] = 0 \tag{2.3}$$
$$\int_{F\times M} d\mu^T |L[u,f,\varphi]|^2 = 0$$
$$\forall\psi \in C\ (F\times M),\quad \forall\varphi\in\Phi$$

This means that the measure μ^T is lumped in solutions of (2.1). The contraction μ_0 of this measure in $C(F)$ is in agreement with the measure induced by the mapping $\omega \to f(|\omega)$. The continuation $f(|\omega)\to u(|\omega)$ of this latter mapping induces a measure that is in agreement with the contraction μ_1 of the measure μ^T in $C(M)$.

By virtue of the conditions used, the random field $f(r)$ is similar to a Gaussian field in the sense of the representation of the higher correlation moments in terms of the lower moments. Let its first moment be zero, and let the second correlation moment $K(r,r')$ be the kernel of a non-degenerate integral operation on the set of functions $W \supset U$ given in the segment $[0,1]$ and satisfying conditions (1.2). It then follows from relationships (1.1), (2.2), (2.3) that

$$\int_M d\mu_1 \int_0^1 dr \int_0^1 dr_1 N[r;u] K^{-1}(r,r_1) N[r_1;u] = \int_0^1 dr\,\delta(r,r_1) = 1 \tag{2.4}$$

Here $\delta(r,r_1)$ is the Dirac δ-function given in the segment $[0,1]$ and related by the conditions (1.2).

We assume that auxiliary measures ν_1 and ν_0 exist in an appropriate manner in the sets M and F such that the measure μ_1 is absolutely continuous with respect to ν_1 and the measure μ_0 with respect to ν_0. According to a well-known Radon theorem, there hence results the existence of Radon-Nikodym derivatives $d\mu_1/d\nu_1$ and $d\mu_0/d\nu_0$. The derivative $d\mu_1/d\nu_1$ is defined for all points of the space M with the exception, perhaps, of a certain subset M_1 for which $\nu_1(M_1)=0$. (By virtue of the condition presented above for absolute continuity, $\mu_1(M_1) = 0$ simultaneously). The derivative $d\mu_0/d\nu_0$ is in turn defined for all points of the space F with the exception, perhaps, of a certain subspace F_1 for which $\nu_0(F_1)=0$.

The measure of uncertainty in the information, the information entropy corresponding to this statistical distribution μ_i $(i=0,1)$, is defined by the formula

$$H_i = -\int_{S_i} d\mu_i \ln\frac{d\mu_i}{d\nu_i} \quad (S_0 = F, S_1 = M)$$

The probabilistic measure μ_1 can be found by the rules of information theory /6,7/ by the Lagrange method from the condition of maximum entropy H_1 simultaneously in combination with the non-negativity and normalizability of μ_1 and the presence of the integral (2.4). Then the absolute extremum of the functional

$$\psi = -\int_M d\mu_1 \Big[\ln\frac{d\mu_1}{d\nu_1} - \beta_1 \int_0^1 dr \int_0^1 dr_1 N[r;u] K^{-1}(r,r_1) N[r_1;u] + \beta_2\Big]$$

corresponds to the conditional extremum of H_1.

The coefficients β_1, β_2 here are Lagrange multipliers. The maximum of ψ is reached in the measure

$$d\mu_1 = c\exp\{-\beta_1 S[u]\}\, d\nu_1;\quad c = \exp(-1-\beta_2) \tag{2.5}$$
$$S[u] = \int_0^1 dr \int_0^1 dr_1 N[r;u] K^{-1}(r,r_1) N[r_1;u]$$

The quantities β_1 and c are given by (2.4) and the condition for normalizability of the measure μ_1.

The invariance of the information entropy required according to information theory postulates /6/ (for the passage from the measure μ_0 to the measure μ_1) under the induced mapping $\omega\to f(|\omega)\to u[f(|\omega)]$ is ensured by selecting ν_1 in the following form:

$$-d\nu_1 = |\operatorname{Det}[\Gamma^{-1}(u)]|\, d\nu_{11} \tag{2.6}$$

Here $d\nu_{11}$ is a genralized measure given on M (by the terminology in /8/) that represents the analogue of the element of "volume" in the space of functions $u(r)$ from M; $\operatorname{Det}[\Gamma^{-1}(u)]$ is the Fredholm determinant of the integral operation in the segment $[0,1]$ with kernel $\Gamma^{-1}[r,r';u]$. The two-point function $\Gamma[r,r';u]$ satisfies the conditions (1.2) and the equation

$$\int_0^1 dr_1 \frac{\delta N[r;u]}{\delta u(r_1)} \Gamma[r_1,r';u] = \delta(r,r')$$

We substitute relationships (2.5), (2.6) into the second integral in (2.4). This integral is then formally reduced by the transformation (1.1) to a Gaussian type continual integral in F

$$c\int_F d\nu_{01} f(r) f(r') \exp\left\{-\beta_1 \int_0^1 dr \int_0^1 dr_1 f(r) K^{-1}(r, r_1) f(r_1)\right\} = K(r, r') \tag{2.7}$$

Here $d\nu_{01}$. the generalized measure in F, is the analogue of the element "volume" in the space of the functions $f(r)$ from F.

Carrying out continual integration in (2.7) by the method in /9/, we obtain that $\beta_1 = {}^1/_2$.

We extract the generalized derivative in relationships (2.5) and (2.6)

$$P_\varepsilon[u] = \frac{d\mu_1}{d\nu_1} = c\,|\operatorname{Det}[\Gamma^{-1}(u)]|\exp\left\{-\frac{1}{2}S[u]\right\} \tag{2.8}$$

The functional $P_\varepsilon[u]$, called later the probability functional (PF), is an infinite-dimensional generalization of the finite-dimensional probability density of the quantities $u(r_i)$ from M into the functional space M (r_i are the coordinates of points of a finite partition of the segment $[0, 1]$).

3. The condition of a local extremum requires disappearance of the variational derivative of the PF $P_\varepsilon[u]$ on the extremals $u = u_* \in M$. The equation for u_* corresponding to this condition has the form

$$N[r; u_*] = -\int_0^1 dr_1 \int_0^1 dr_2 K(r, r_1) \frac{\delta}{\delta u(r_2)} \Gamma[r_2, r_1; u]\Big|_{u=u_*} \tag{3.1}$$

A non-analytic operation with kernel $\Gamma[r, r'; u = u_*]$ is on the right side of (3.1). However, it follows from (2.8) that degeneration of the matrix $\Gamma^{-1}[r, r'; u = u_*]$ sets in only in those realizations of its functional argument u_* from U for which the probability is zero. Therefore, the operators whose kernels comprise the matrix of $\Gamma[r, r'; u = u_*]$, are analytic, in probability, with respect to the functional argument u_* or other parameters of the problem.

Using this circumstance, we transform (3.1) to a form containing only lower-order terms in the geometric non-linearity parameter α. We set $\lambda = \lambda_0 + \gamma$. In a small neighbourhood of λ_0 let

$$\operatorname{Det}[\Gamma^{-1}(u)]|_{u=0} \neq 0$$

Then expansion of the components on the right side of (3.1) in powers of α is possible with non-zero probability. Taking account just of the principal terms in this expansion, we obtain (introducing the notation $\chi = u_*(r)$)

$$B(r; \lambda_0)\chi = -\gamma\chi - \alpha N_1[r; \chi] - \alpha\varepsilon N_2[r; \chi] + \alpha\psi_1(r; \lambda, \lambda_0)\chi + \alpha\varepsilon\psi(r; \lambda) + o(\alpha) \tag{3.2}$$

$$B(r; \lambda_0)\chi = L\chi|_{\lambda=\lambda_0} - \alpha\int_0^1 ds M(r, s; \lambda_0)\chi(s)$$

$$M(r, s; \lambda) = 2\int_0^1 ds_1 \int_0^1 dr_1 K(r, s) g(r_1, s)(r_1 s)^{-3} \times$$
$$[2\Gamma_0(s, r_1; \lambda)\Gamma_0(r_1, s_1; \lambda) + \Gamma_0(r_1, r_1; \lambda)\Gamma_0(s, s_1; \lambda)]$$

$$\psi(r; \lambda) = 2\int_0^1 ds \int_0^1 ds_1 \int_0^1 dr_1 K(r, s) g(r_1, s_1)\Gamma_0(r_1, s; \lambda) \times$$
$$[2r_1^{-3}\Gamma_0(s_1, r_1; \lambda) + s_1^{-3}\Gamma_0(s_1, s_1; \lambda)]$$

$$\psi_1(r; \lambda, \lambda_0)\chi = \int_0^1 ds[M(r, s; \lambda) - M(r, s; \lambda_0)]\chi(s)$$

Here $B(r; \lambda_0)$ is a linear operator; the function $\Gamma_0(r, s; \lambda)$ equals the functions $\Gamma[r, r'; u = 0]$ at the values $\lambda > 0$, and in the general case is bounded in probability.

Study of the solutions of (3.2) with analytic operators, in probability, the clarification of the moments of their bifurcations, the extraction from these of those that correspond to generation and mutual transitions of the modes of the probability functional, comprise the content of a stability investigation (in the ensemble sense) of this and kindred /1/ stochastic systems.

Let $\zeta_m(r)$ denote the zeros of the operator $B(r; \lambda_0)$. Under conditions (1.2) they will be non-trivial solutions of the linear equation

$$B(r; \lambda_0)\chi = 0 \tag{3.3}$$

We let λ_{0m} denote values of the quantity λ_0 corresponding to moments of irreversibility of the operator $B(r; \lambda_0)$.

We examine the case when just one zero of the operator $B(r;\lambda_0)$ corresponds to each value of $\lambda_{0m} > 0$. In this case (3.2) allows a formal solution of the form (analogous to non-linear equations with operators in Banach spaces /10/)

$$\chi(r) = \eta\zeta_m(r) - \alpha\int_0^1 ds R_m(r,s)[\alpha^{-1}\gamma\chi(s) + N_1[s;\chi] - \psi_1(s;\lambda,\lambda_0) + \varepsilon(N_2[s;\chi] - \psi(s;\lambda))] + o(\alpha) \tag{3.4}$$

$$\eta = \int_0^1 dr\chi(r)\zeta_m(r) \tag{3.5}$$

Here $R_m(r,s)$ is the kernel of Green's operator determined under conditions (1.2) by the expression

$$B(r;\lambda_{0m})R_m(r,r') + \int_0^1 ds\zeta_m(r)\zeta_m(s)R_m(s,r') = \delta(r,r')$$

4. We consider the case when $\varepsilon = 0$ (circular membranes). In this case (3.1) has both zero and non-zero solutions. We pose the following question: for what values of λ is the trivial extremal $u_* = 0$ a mode of the probability functional of the membranes $P_0[u] = P_{\varepsilon=0}[u]$.

Analogous to non-linear equations with bounded operators with probability one /10/, the solution of (3.4) is representable as the following convergent series in probability as $\varepsilon = 0$:

$$\chi(r) = \sum_{k=1} \chi_k(r)\eta^k \tag{4.1}$$

The quantities χ_k are found by the method of undetermined coefficients, i.e. by substituting (4.1) into (3.4) and subsequent comparison of the coefficients of identical powers of η. The so-called bifurcation equation for the quantity η is formed by substituting (4.1) into (3.5). By carrying out these operations we obtain

$$L_{11}(\lambda - \lambda_{0m}) - \alpha L_{12}(\lambda,\lambda_{0m}) + \alpha L_3\eta^2 \approx 0 \tag{4.2}$$

$$L_{11} = \int_0^1 dr\zeta_m^2(r),\quad L_{12}(\lambda,\lambda_{0m}) = \int_0^1 dr\zeta_m(r)\psi_1(r;\lambda,\lambda_0)\zeta_m$$

$$L_3 = \int_0^1 dr\int_0^1 ds(rs)^{-1/2}g(r,s)\zeta_m^2(r)\zeta_m^2(s)$$

As is clear from (3.2) and (4.2), the following are approximate small solutions of Eqs. (3.1):

$$u_*(r) \approx u_\pm(r,\lambda-\lambda_{0m}) = \pm\beta\zeta_m(r) \tag{4.3}$$

$$\beta = [\alpha^{-1}|L^{(1)}(\lambda,\lambda_{0m})L_3^{-1}|(\lambda-\lambda_{0m})]^{1/2}$$

$$L^{(1)}(\lambda,\lambda_{0m}) = L_{11} - \alpha L_{12}(\lambda,\lambda_{0m})(\lambda-\lambda_{0m})^{-1}$$

It hence follows that (3.1) has just a trivial solution in the domain $0 < \lambda < p_{0n}$ (p_{0n} is the least quantity from the sequence $\{\lambda_{0m}\}$). This means that here the probability functional $P_0[u]$ reaches the extremum only in the mode $u(r) = 0$. The nature of this extremum is established from the continuity (in probability) of the functional $P_0[u]$ and the conditions

$$P_0[u] \geqslant 0,\ P_0[u=0] > 0,\ P[u \to \pm\infty] \to 0 + 0 \tag{4.4}$$

These conditions indicate that the extremal $u_*(r) = 0$, which is unique here, is a mode of the functional $P_0[u]$.

The second variation of the functional $S_1[u] = -\ln(c^{-1}P_0[u])$ has the form of a simple quadratic form in the neighbourhood of $u = 0$

$$\delta^2 S_1[u] = \frac{1}{2}\sum_m c_m(\lambda)u_m^2,\quad u_m = \int_0^1 dr\zeta_m(r)u(r) \tag{4.5}$$

The quantities $c_m(\lambda)$ are the eigennumbers of the operator on the left side of the equation

$$\int_0^1 dr_1\frac{\delta^2 S_1[u]}{\delta u(r_1)\delta u(r)}\bigg|_{u=0} u(r_1) = cu(r) \tag{4.6}$$

The solutions of (4.6) agree, in probability, with the zeros $\zeta_m(r)$ of the operator $B(r,\lambda)$.

The probability functional $P_0[u]$ has just one extremum in the domain $0 < \lambda < p_{0n}$ the maximum in the function $u(r) = 0$. Therefore, the quantity $S_1[u]$ is here a functional convex downward $\delta^2 S_1[u] > 0$ in the small neighbourhood $u(r) = 0$ and all $c_m(\lambda) > 0$. Upon achievement (and a small subsequent increase) by the compressive force λ of the value p_{0n}, the eigennumber $c_n(\lambda)$ passes through zero into the domain of negative values. The quadratic form (4.5) becomes sign-definite. Therefore, the trivial solution of (3.1) for $\varepsilon = 0$ ceases to be a mode of the functional $P_0[u]$. By virtue of conditions (4.4) modes for the functional $P_0[u]$ should be here among the remaining solutions of (3.1). Within the framework of the approximations used, there are two such solutions in a sufficiently small positive semicircle of p_{0n}: $u_\pm(r; \lambda - p_{0n})$. Since $P_0[u] = P_0[-u]$, both these solutions will approximate modes of the membrane probability functional.

Therefore, the membrane probability functional $P_0[u]$ has a single extremum up to the time $\lambda = p_{0n}$ of the first bifurcation of the trivial solution of the extremal equation, the maximum on $u(r) = 0$ which goes over into new maxima at approximately the modes $u_\pm(r, \lambda - p_{0n})$ at the time noted.

Such a phenomenon is an example of stochastic bifurcation.

For sufficiently small $K(r, r')$ problem (3.3), (1.2) allows the approximate solutions $\zeta_m(r) \approx \varphi_m(r) = (2r)^{1/2} |J_2(k_m)|^{-1} J_1(k_m r)$ within the framework of the Galerkin method ($J_n(x)$ are n-th order Bessel functions of the first kind, and k_m are the roots of the equation $J_1(x)$, $x > 0$). The values λ_{0m} of the force λ corresponding to times of irreversibility of the operator $B(r, \lambda)$ are found in the neighbourhoods of the quantities k_m^2 in this case. Approximately

$$\lambda_{0m} \approx -q_m^2 = k_m^2 - (6\alpha\beta_m K_{mm})^{1/2} \tag{4.7}$$

$$K_{pp} = \int_0^1 dr \int_0^1 dr_1 \varphi_p(r) K(r, r_1) \varphi_p(r_1)$$

$$\beta_p = \sum_n k_n^{-2} \left(\int_0^1 ds s^{-1} \varphi_n(s) \varphi_p^2(s) \right)^2$$

According to (4.3), the following will be approximate small solutions of (3.1) for $\varepsilon = 0$:

$$u_\pm^{(m)}(r; \lambda, q_m^2) = \pm a_m(\lambda) [(\alpha\beta_m)^{-1}(\lambda - q_m^2)]^{1/2} \varphi_m(r) \tag{4.8}$$
$$a_m(\lambda) = |\lambda - k_m^2|^{-1} [(\lambda - q_m^2)^2 + 3(\lambda - q_m^2)(q_m^2 - k_m^2) + 3(q_m^2 - k_m^2)^2]^{1/2}$$

In the case $K = 0$, expressions (4.7) and (4.8) determine the known /10/ sequence $\{k_m^2\}$ of equilibrium bifurcation points of membranes with zero transverse load and derivatives of the equilibrium deflection modes $v_\pm^{(m)}(r)$

$$\frac{dv_\pm^{(m)}}{dr} \approx \pm [(\alpha\beta_m)^{-1}(\lambda - k_m^2)]^{1/2} |J_2(k_m)|^{-1} J_1(k_m r)$$

Let the correlation moment $B(r, r')$ of a random field of dimensionless transverse loads $\pi(r)$ be approximated by the δ-functions

$$B(r, r') = A\delta(r, r') \tag{4.9}$$

The correlation moment of the field $f(r)$ corresponding to this approximation has the following form:

$$K(r, r') = A \min(r, r')(rr')^{-1/2} \tag{4.10}$$

Computations performed by means of (4.7) and (4.10) show that over a fairly broad range of variation of the amplitude $A = [0, 10]$ in which the approximations used are valid, the least quantity from the sequence $\{q_m^2\}$ is reached at the value $m = 1$, i.e.,

$$q_1^2 = \min\{q_m^2\} \approx k_1^2 - 0.69(\alpha A)^{1/2}, \quad k_1^2 \approx 14.69 \tag{4.11}$$

The case similar to (4.11) (in the sense of the existence of a minimum of the sequence $\{q_m^2\}$ at the value $m = 1$) holds for a field of transverse loads with a non-zero radius of correlation and the moment

$$B(r, r') = A \exp\{-|r - r'| r_0^{-1}\}, \quad 0 < r_0 \ll 1 \tag{4.12}$$

Here r_0 is the scale correlation parameter.

In both cases (4.9) and (4.12), the stochastic bifurcation of the membrane equilibrium sets in earlier (as the compressive force increases) than the deterministic bifurcation of a membrane with zero transverse load.

The influence of the stochastic loads on the capacity to buckle is estimated by the magnitude of the relative distance γ_0 between the first (in the compressive force) point of bifurcation of the equilibrium of a membrane with zero transverse load and stochastic bifurcation point. In this case

$$\gamma_0 = (k_1^2 - q_1^2) k_1^{-2} \tag{4.13}$$

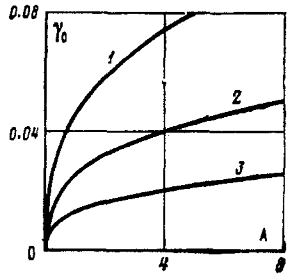

Fig.1

Computed dependences of the quantity γ on the amplitude A of the correlation moments of a random field of transverse loads are represented in Fig.1 for the case $\alpha = 0.1$. Curve *1* corresponds to a δ-correlated field of loads (the moment (4.9)). Curves *2, 3* correspond to load fields with correlation moments of the type (4.12) with the scale parameters $r_0 = 0.1$ and $r_0 = 0.01$, respectively. It follows from the shape of these curves that the predisposition to buckling of the intial mode is raised markedly for membranes with a stochastic transverse load compared with membranes with zero transverse load even in the domain $A < 1$.

5. We examine the case $0 < \varepsilon \ll 1$ (shallow shells). For small ε and α problem (3.2), (1.2) allows the following approximate solution within the framework of the Galerkin method:

$$\chi(r) \approx \eta \zeta_m(r) \tag{5.1}$$

The parameter η in (5.1) is subject to the equation

$$\alpha\varepsilon L_0 + [L_{11}(\lambda - \lambda_{0m}) - \alpha L_{12}(\lambda, \lambda_{0m})]\eta + \alpha\varepsilon L_2 h^2 + \alpha L_3 \eta^3 \approx 0 \tag{5.2}$$

The coefficients of (5.2) satisfy the relationships (4.2) and the equalities

$$L_0 = \int_0^1 dr \zeta_m(r)\psi(r, \lambda), \quad L_2 = \int_0^1 dr \int_0^1 ds r^{-1/2} \zeta_m^2(r) g(r, s) \zeta_m(s)$$

Up to the time $\lambda = \lambda_{1m}$ Eq.(5.2) has one, non-trivial, solution η_0. Starting with this time there will be three such solutions: η_0, η_+, η_-. Within the framework of the approximation $\zeta_m(r) \approx \varphi_m(r)$ we obtain

$$\lambda_{1m} \approx \mu_m^2 = q_m^2 + (2^{-1}6^{1/3}\varepsilon\alpha^{1/3}\tau_m\beta_m^{-1/3}K_{mm})^{2/3} \tag{5.3}$$

$$\gamma_m = k_m^{-2}\int_0^1 ds s^{-3/2}\varphi_m^3(s)$$

In the neighbourhoods of the quantities $\lambda = \lambda_{1m}$ the behaviour of the amplitudes $\eta_0, \eta_\pm$ is characterized by the following dependence on λ:

$$\eta_0 \sim (4 - g_m(\lambda))^{1/3} - y_m \tag{5.4}$$

$$\eta_\pm \sim -g_m^{1/3}(\lambda)[2^{-1/3} \pm (-4^{-1/3} + 3^{-1}d_m(\lambda) g_m^{-2/3}(\lambda))^{1/2}] - y_m \tag{5.5}$$

$$y_m = 2\varepsilon(3\beta_m)^{-1}\tau_m, \delta_m = \lambda - \lambda_{0m}, d_m(\lambda) = -3y_m^2 - (\alpha\beta_m)^{-1}\delta_m$$

$$g_m(\lambda) = y_m[(\alpha\beta_m)^{-1}\delta_m - (6\alpha^{-2}\beta_m^{-1}K_{mm})^{1/3}]$$

For large absolute values of the quantities $\xi_m = (\alpha\beta_m)^{-1}\delta_m$ the asymptotic form of the solutions of (5.2) acquires the following form

$$\eta_0 \sim \begin{cases} \eta_{01} = -6\varepsilon\tau_m K_{mm}\beta_m^{-1}(\lambda - \lambda_{0m})^{-3}, & \xi_m \to -\infty \\ \eta_{02} = \operatorname{sign}(g_m(\lambda))\xi_m^{1/2}, & \xi_m \to \infty \end{cases} \tag{5.6}$$

$$\eta_+ \sim -\eta_{02}, \quad \eta_- \sim -\eta_{01}, \quad \xi_m \to \infty \tag{5.7}$$

We extract the least quantity λ_{1k} from the sequence $\{\lambda_{1m}\}$. We assume that just one pair $(\zeta_k(r), \lambda_{0k})$ corresponds to it. Then, to a first approximation, (3.1) has simple solutions of the type (5.1) for small ε, α

$$\chi(r) \approx \eta\varphi_k(r) \tag{5.8}$$

Substituting here (5.4)-(5.7) instead of η, we obtain the asymptotic form of small solutions of the extremal equation for the shell PF $P_\varepsilon[u]$ before and after the first point of their bifurcation.

The functional $P_\varepsilon[u]$ is continuous in probability in the set M. It satisfies the conditions

$$P_\varepsilon[u] \geqslant 0 \ (\forall u \in M), \ P_\varepsilon[u \rightrightarrows {}^{+\infty}_{-\infty}] \to 0 + 0 \tag{5.9}$$

It follows from these conditions that within the framework of the approximations utilized the single extremal $\chi(r) \approx \eta_0\varphi_k(r)$ is the mode of the PF $P_\varepsilon[u]$ in the domain $0 < \lambda < \mu_k^2 = \min\{\mu_m^2\}$.

Approximately from the time $\lambda = \mu_k^2 = \min\{\mu_m^2\}$ two new non-trivial solutions appear for the extremal Eq.(3.1) (for $\varepsilon > 0$): $\chi_\pm(r) = \eta_\pm\varphi_k(r)$. Both these solutions start from one point in the (λ, η) plane with coordinates

$$\lambda = \mu_k^2, \quad \eta \sim -(2^{-1} g_k (\lambda = \mu_k^2))^{1/2} - 2\varepsilon (3\beta_k)^{-1} \tau_k$$

It follows from (5.4) and (5.8) that in the domain $0 < \lambda < \mu_k^2$ the curve $\eta_-(\lambda)$ lies between the curves $\eta_0(\lambda)$ and $\eta_+(\lambda)$ in the (λ, η) plane. (Note that as $\xi_k \to \infty$ the amplitudes η_+, η_0 differ only in sign, while the quantity η_- is proportional to the small ratio ε/α for $\varepsilon \ll \alpha$). Hence, and from conditions (5.9) it follows that within the framework of the approximations used, the functional $P_\varepsilon[u]$ has maxima in the modes $\chi_0(r) = \eta_0 \varphi_k(r)$, $\chi_+(r) = \eta_+ \varphi_k(r)$ in the domain $\lambda > \mu_k^2$. The time of origination of the second maximum for $\lambda = \mu_k^2$ is accompanied by the inequality $P_\varepsilon[\chi_+] < P_\varepsilon[\chi_0]$, which is replaced by the approximate equality $P_\varepsilon[\chi_+] \lessapprox P_\varepsilon[\chi_0]$ as $\xi_k \to \infty$.

Therefore, for a given stochastic elastic system (an open shallow spherical shell of circular outline with a random Gaussian transverse load and uniform radial stress on the contour), the most probable deflection mode in the domain $\lambda < \mu_k^2 = \min\{\mu_m^2\}$ is the single mode $v_0(r)$ for which

$$\frac{dv_0(r)}{dr} \approx \eta_0 I_k(r), \quad I_k(r) = |J_2(k_k)|^{-1} J_1(k_k r)$$

In the domain $\xi_k \to \infty$ the most probable deflection modes are two: $v_0(r)$ and $v_+(r)$, where

$$\frac{dv_0(r)}{dr} \sim \eta_{02} I_k(r), \quad \frac{dv_+(r)}{dr} \sim -\eta_{02} I_k(r)$$

Let V_* be the set of possible shell deflection modes, and V_0, V its subsets which are non-intersecting neighbourhoods of the elements $v_0 \in V_0, v_+ \in V_+$ from V_*, respectively. The presence of two equilibrium deflection modes $v_0(r), v_+(r)$ as $\xi_k \to \infty$ that have almost equal maximum probabilities of existence on V (replaced by one maximally probable mode $v_0(r)$ for $\lambda < \mu_k^2$) generates a moderate probability of transition from one of the elements of the set V_0 to one of the elements of the set V_+ for small system perturbations in the case $\lambda > \mu_k^2$. Such a phenomenon is a case of stochastic snap-through.

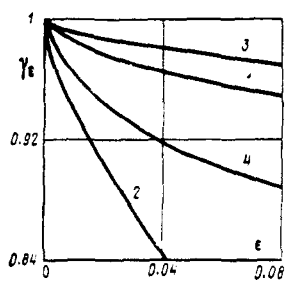

Fig.2

For transverse loads with the correlation moments (4.9) and (4.12), the least value from the sequence $\{\mu_m^2\}$ is reached in the domain $A = [0, 10]$ at the value $m = 1$, i.e., the equality $\mu_1^2 = \min\{\mu_m^2\}$ holds. The influence of the parameter ε on the stochastic bifurcation can be estimated from the relationship

$$\gamma_\varepsilon = \frac{k_1^2 - \mu_1^2}{k_1^2 - q_1^2}$$

Computed dependences of the quantity γ_ε on ε for different values of the correlation-moment amplitudes are given in Fig.2. Curves 1 and 3 correspond to δ-correlated fields of transverse loads with amplitudes $A = 0.1$ and $A = 9.4$, respectively. Curves 2 and 4 correspond to fields of transverse loads with moments of the type (4.12) for one common value of the parameter $r_0 = 0.1$ and two values $A = 0.1$ and $A = 9.4$, respectively. The behaviour of the curves in Fig.2 indicates that the initial shell curvature exerts a retarding influence on the possibility of stochastic bifurcation. In other words, as ε grows, more and more compressive force is required so that two maxima appear in the probability functional instead of one.

REFERENCES

1. VOLKOV S.I., Example of stochastic bifurcation in the theory of non-ideal plate bending. PMM, 45,5,1981.
2. EDWARDS S.F., The statistical dynamics of homogeneous turbulence. J. Fluid Mech., 18 1964.
3. FEODOS'EV V.I., Elastic Elements of Precision Instrumentation. Oboronizdat, Moscow, 1949.
4. ARSEN'EV A.A., On the existence of stochastic and statistical solutions of non-linear equations. Problems of Mathematical Physics and Computational Mathematics, Nauka, Moscow, 1977.
5. BOURBAKI N., Elements of Mathematics. Book 6. Measure, Integration of Measure./Russian translation/, Nauka, Moscow, 1967.
6. STRATONOVICH R.L., Information Theory. Sovetskoe Radio, Moscow, 1975.
7. ZUBAREV D.N., Non-equilibrium Statistic Thermodynamics. Nauka, Moscow, 1971.
8. GEL'FAND I.M. and VILENKIN N.YA., Generalized Functions. Fizmatgiz, Moscow, 1959.
9. SLAVNOV A.A., The continual integral in perturbation theory. Teoret. Matem. Fizika, 22, 2, 1975.

10. VAINBERG M.M. and TRENOGIN V.A., Theory of the Bifurcation of Solutions of Non-linear Equations. Nauka, Moscow, 1969.

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ASYMPTOTIC FORM OF THE STRESS INTENSITY COEFFICIENTS IN QUASISTATIC TEMPERATURE PROBLEMS FOR A DOMAIN WITH A CUT*

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Plane quasistatic thermoelasticity problems are investigated for domains of arbitrary shape with a cut in the case of an instataneous temperature change on the boundary. The asymptotic form of the stresses is investigated in the neighbourhood of a crack tip.

Certain quasistatic temperatue problems were solved earlier in /1-5/ (see /6/ also) for the development of cracks on parts of whose surfaces a constant temperature occurs at the initial instant and is maintained. Expressions are obtained for the stress intensity coefficients at the crack tip.

Quasistationary thermoelasticity problems are investigated below for domains with cut in a more general asymptotic sense. A plane domain with a cut whose boundary is instantaneously cooled or heated is examined in Sects.1−3. Since the shape of the domain contour can be arbitrary, it is impossible to speak of the explicit solution of the thermoelasticity boundary value problem. Nevertheless, an expression is successfully found for the principal terms of the asymptotic form of the stress intensity coefficients at the most dangerous initial times (from the viewpoint of crack propagation). In particular, the asymptotic form of the fracture time is determined as a function of the temperature jump at the crack tip.

Note that the principal term of the tensile stress intensity coefficient is independent of the contour shape, and agrees with the intensity coefficient of the same problem for a plane with a cut.

Analogous results are obtained in Sec.4 for the problem of an instantaneous change in the endface temperature of a thin plate from whose side surfaces heat is transferred to the external medium, where the stress intensity coefficients found are explicitly expressed in terms of those in the absence of heat transfer. This enables an asymptotic analysis to be made of the stresses near a crack tip at the initial times.

The results obtained in this paper emerge from the asymptotic solution of the heat conduction equations as $t\to 0$ for a domain with a cut and the method proposed in /7/ for calculating the stress intensity coefficients.

1. Formulation of the boundary value problems. To be specific we will examine plane strain. As is well-known, the plane state of stress with zero heat transfer from the external medium is realized on replacing the Lamé constant λ by $\lambda_* = 2\lambda\mu\,(\lambda+2\mu)$, and γ by $\gamma_* = (1-2\nu)\,\gamma\,(1-\nu)$, where $\gamma = 2\mu\alpha_T\,(1+\nu)/(1-2\nu)$; μ is the shear modulus, α_T is the coefficient of linear expansion, and ν is Poisson's ratio, in which connection, only the appropriate constants vary inthe asymptotic formulas indicated later.

Let Ω_0 be a plane domain with a smooth boundary Γ_0 (see the Figure). There is a rectilinear cut of length l in Ω_0 that connects the origin $O\in\Omega_0$ with the point $A\in\Gamma_0$. We denote the upper and lower edges of the cut by l_+ and l_-. We understand Γ to be the contour Γ_0 supplemented with two drawn segments l. and Ω to be the domain bounded by Γ Let Ω^c be the closure of the domain Ω in the sense of its internal metric. To simplify the discussion, we will consider the angle formed by the contour Γ_0 and the segment l to be a right angle, and the contour Γ_0 itself to be rectilinear near the point A.

The temperature T is determined from the solution of the boundary value problem

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